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Approximate potential symmetries for partial differential equations

A H Kara†, F M Mahomed‡ and Changzheng Qu§

† Department of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa
‡ Department of Computational and Applied Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
§ Department of Mathematics, Northwest University, Xi'an 710069, People's Republic of China

E-mail: kara@cam.wits.ac.za, fmahomed@cam.wits.ac.za and qu-changzheng@hotmail.com

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Abstract. The method of approximate potential symmetries for partial differential equations with a small parameter is introduced. By writing a given perturbed partial differential equation R in a conserved form, an associated system S with potential variables as additional variables is obtained. Approximate Lie point symmetries admitted by S induce approximate potential symmetries of R. As applications of the theory, approximate potential symmetries for a perturbed wave equation with variable wave speed and a nonlinear diffusion equation with perturbed convection terms are obtained. The corresponding approximate group-invariant solutions are also derived.

1. Introduction

Group-theoretical methods based on local symmetries are useful and effective to construct group-invariant solutions, to calculate conservation laws of partial differential equations (PDEs) [6, 17, 18, 23, 24, 26] and to linearize nonlinear PDEs by invertible mappings [6, 7]. Local symmetries include Lie point symmetries [6,17,23,24,26], contact symmetries [6,17,23, 24, 26], conditional symmetries (or the nonclassical method for symmetries) [8], generalized symmetries [6,17,23] and more general generalized conditional symmetries [14,25,28]. These methods have been successfully applied to obtain exact solutions and find conservation laws of PDEs.

For a given system of PDEs one could also find useful nonlocal symmetries through embedding it in an auxiliary covering system with additional dependent variables [1, 6, 9–11, 20, 27]. A Lie point symmetry of the auxiliary system, acting on the space consisting of the independent and dependent variables of the given system as well as the auxiliary variables, yields a nonlocal symmetry of the given system if it does not project to a point symmetry acting on its space of independent and dependent variables. Such an auxiliary system is obtained by the replacement of a PDE of the given system by an equivalent conservation law. Frequently, the corresponding nonlocal symmetries are called potential symmetries of the given system.

Another vital aspect is that many PDEs in applications depend on a small parameter, so it is of great importance and interest to find approximate solutions and conservation laws. Baikov *et al* [2–5], in a series of papers, developed a theory and applications of the approximate symmetry group method to find approximate invariant solutions, to calculate approximate conservation laws (see also [19]) and approximate Lie–Bäcklund transformation groups of nonlinear PDEs. In [21], the approximate conditional symmetry method was introduced and was used to construct new approximate conditional symmetries and new approximate solutions of a perturbed wave equation and nonlinear diffusion equations with source terms.

The purpose of this paper is to present a method of approximate potential symmetries for PDEs. The outline is as follows. In section 2, we discuss the method of approximate potential symmetries. In sections 3 and 4, we use the method of section 2 to obtain approximate potential symmetries for a wave equation with variable wave speed and a nonlinear diffusion equation with a perturbed convection term. In section 5, we present examples to show how one can obtain approximate invariant solutions, of up to order ϵ (ϵ is a small parameter), by the method of section 2. Section 6 presents a summary of our results.

2. Approximate potential symmetries of PDEs

Consider a scalar kth-order perturbed PDE $R\{x, u, \epsilon\}$, which is written in a conserved form

$$D_i[f^i(x, u, u_{(1)}, \ldots, u_{(k-1)}) + \epsilon g^i(x, u, u_{(1)}, \ldots, u_{(k-1)})] = 0 \tag{1}$$

with $n \geqslant 2$ independent variables $x = (x_1, \ldots, x_n)$, a single dependent variable u, $u_{(j)}$, $j = 1, \ldots, k-1$, is a collection of jth-order partial derivatives: $u_{(1)} = (u_1, \ldots, u_n)$, $u_{(2)} = (u_{11}, u_{12}, \ldots, u_{nn}), \ldots$, where ϵ is a small parameter and

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{ii_1 i_2,\ldots,i_{k-1}} \frac{\partial}{\partial u_{i_1 i_2,\ldots,i_{k-1}}} \qquad i = 1, 2, \ldots, n.$$

Since the PDE (1) is in a conserved form, there exist $\frac{1}{2}n(n-1)$ functions $\psi^{i,j}$ components of an antisymmetric tensor ($i < j$), such that (1) can be expressed in the form

$$\begin{aligned} &f^i(x, u, u_{(1)}, \ldots, u_{(k-1)}) + \epsilon g^i(x, u, u_{(1)}, \ldots, u_{(k-1)}) \\ &\qquad = \sum_{i<j}(-1)^j \frac{\partial}{\partial x_j}\psi^{i,j} + \sum_{j<i}(-1)^{i-1}\frac{\partial}{\partial x_j}\psi^{j,i} \qquad i, j = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

Setting

$$\psi^{i,j} = 0 \qquad \text{for} \quad j \neq i+1$$

and introducing

$$v^i = \psi^{i,i+1} \qquad i = 1, 2, \ldots, n-1$$

the system (2) associated with $R\{x, u, \epsilon\}$ given by (1) becomes the following auxiliary system of PDEs: $S\{x, u, v, \epsilon\}$

$$\begin{aligned} f^1 + \epsilon g^1 &= \frac{\partial}{\partial x_2} v^1 \\ f^j + \epsilon g^j &= (-1)^{j-1}\left[\frac{\partial}{\partial x_{j+1}} v^j + \frac{\partial}{\partial x_{j-1}} v^{j-1}\right] \qquad 1 < j < n \\ f^n + \epsilon g^n &= (-1)^{n-1}\frac{\partial}{\partial x_{n-1}} v^{n-1}. \end{aligned} \tag{3}$$

For $n = 2$, let

$$\begin{aligned} f^1 + \epsilon g^1 &= f(x, u, u_{(1)}, \ldots, u_{(k-1)}) + \epsilon g(x, u, u_{(1)}, \ldots, u_{(k-1)}) \\ f^2 + \epsilon g^2 &= -p(x, u, u_{(1)}, \ldots, u_{(k-1)}) - \epsilon q(x, u, u_{(1)}, \ldots, u_{(k-1)}) \end{aligned} \tag{4}$$

so that $R\{x, u, \epsilon\}$ becomes

$$D_1(f + \epsilon g) - D_2(p + \epsilon q) = 0. \tag{5}$$

By the introduction of the potential variable $v = v^1 = \psi^{1,2}$, the auxiliary system $S\{x, u, v, \epsilon\}$ is written as

$$\frac{\partial v}{\partial x_2} = f + \epsilon g \qquad \frac{\partial v}{\partial x_1} = p + \epsilon q. \tag{6}$$

Assume now that the system $S\{x, u, v, \epsilon\}$ given by (6) admits an approximate Lie point symmetry

$$\begin{aligned} V = V_0 + \epsilon V_1 = {} & \xi_{01}(x, u, v)\frac{\partial}{\partial x_1} + \xi_{02}(x, u, v)\frac{\partial}{\partial x_2} + \phi_0(x, u, v)\frac{\partial}{\partial u} + \psi_0(x, u, v)\frac{\partial}{\partial v} \\ & + \epsilon\left[\xi_{11}(x, u, v)\frac{\partial}{\partial x_1} + \xi_{12}(x, u, v)\frac{\partial}{\partial x_2} + \phi_1(x, u, v)\frac{\partial}{\partial u} + \psi_1(x, u, v)\frac{\partial}{\partial v}\right] \end{aligned} \tag{7}$$

which can be calculated by a three-step approach [2, 3, 5]. See below.

(1) Find the Lie point symmetry generators V_0 of the unperturbed equation of (6)

$$\frac{\partial v}{\partial x_2} = f \qquad \frac{\partial v}{\partial x_1} = p \tag{8}$$

by solving the determining equations for the exact symmetries

$$V_0^{[k-1]}\left(\frac{\partial v}{\partial x_2} - f\right) = V_0^{[k-1]}\left(\frac{\partial v}{\partial x_1} - p\right) = 0 \quad \text{satisfied on the equations.}$$

(2) Given V_0 and the perturbation terms g and q, calculate the auxiliary functions H_1 and H_2:

$$H_1 = \frac{1}{\epsilon}V_0^{[k-1]}\left(\frac{\partial v}{\partial x_2} - f - \epsilon g\right)\bigg|_{\{\frac{\partial v}{\partial x_2} = f+\epsilon g,\, \frac{\partial v}{\partial x_1} = p+\epsilon q\}}$$

$$H_2 = \frac{1}{\epsilon}V_0^{[k-1]}\left(\frac{\partial v}{\partial x_1} - p - \epsilon q\right)\bigg|_{\{\frac{\partial v}{\partial x_2} = f+\epsilon g,\, \frac{\partial v}{\partial x_1} = p+\epsilon q\}}.$$

(3) Find the first-order deformations from the determining equations of the deformations

$$\begin{aligned} & V_1^{[k-1]}\left(\frac{\partial v}{\partial x_2} - f\right)\bigg|_{\{\frac{\partial v}{\partial x_2} = f,\, \frac{\partial v}{\partial x_1} = p\}} + H_1 = 0 \\ & V_1^{[k-1]}\left(\frac{\partial v}{\partial x_1} - p\right)\bigg|_{\{\frac{\partial v}{\partial x_2} = f,\, \frac{\partial v}{\partial x_1} = p\}} + H_2 = 0. \end{aligned} \tag{9}$$

If the infinitesimals $(\xi_{01}, \xi_{02}, \xi_{11}, \xi_{12}, \phi_0, \phi_1)$ depend on v explicitly, the generator (7) defines a nontrivial approximate potential symmetry of $R\{x, u, \epsilon\}$. This symmetry is an approximate nonlocal symmetry since the potential variable v defined by the auxiliary system appears in (6). This leads to the following definition.

Definition 1. *The approximate point symmetry (7) admitted by the auxiliary system of PDEs $S\{x, u, v, \epsilon\}$ defines an approximate potential symmetry admitted by $R\{x, u, \epsilon\}$ if and only if $(\xi_{01}, \xi_{02}, \xi_{11}, \xi_{12}, \phi_0, \phi_1)$ depend explicitly on v.*

Suppose that $R\{x, u, \epsilon\}$ is a scalar perturbed evolution equation with two independent variables $x = (x_1, x_2)$ written in conserved form

$$D_2 u = D_1[f(x, u, u_{(1)}, \ldots, u_{(k-1)}) + \epsilon g(x, u, u_{(1)}, \ldots, u_{(k-1)})]. \tag{10}$$

Table 1. Potential symmetries of equation (13).

$c(x)$	Generators
x^a $(a \neq 0, 1)$	$V_0^1 = x\frac{\partial}{\partial x} + (1-a)t\frac{\partial}{\partial t} - av\frac{\partial}{\partial v}$
	$V_0^2 = 2tx\frac{\partial}{\partial x} + \left[(1-a)t^2 + \frac{1}{1-a}x^{2-2a}\right]\frac{\partial}{\partial t} + [(2a-1)tu - xv]\frac{\partial}{\partial u} - (tv + x^{1-2a}u)\frac{\partial}{\partial v}$
x	$V_0^3 = 2tx\frac{\partial}{\partial x} + 2\ln x\frac{\partial}{\partial t} + (tu - xv)\frac{\partial}{\partial u} - (tv + x^{-1}u)\frac{\partial}{\partial v}$
e^x	$V_0^4 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}$
	$V_0^5 = -4t\frac{\partial}{\partial x} + 2(t^2 + \mathrm{e}^{-2x})\frac{\partial}{\partial t} + 2(v - 2tu)\frac{\partial}{\partial u} + 2\mathrm{e}^{-2x}u\frac{\partial}{\partial v}$
$\tilde{c}(x)$	$V_0^6 = \mathrm{e}^t\left[\frac{2\tilde{c}}{\tilde{c}'}\frac{\partial}{\partial x} + 2\left(\left(\frac{\tilde{c}}{\tilde{c}'}\right)' - 1\right)\frac{\partial}{\partial t} + \left(\left(2 - \left(\frac{\tilde{c}}{\tilde{c}'}\right)'\right)u - \frac{\tilde{c}}{\tilde{c}'}v\right)\frac{\partial}{\partial u} - \left(\left(\frac{\tilde{c}}{\tilde{c}'}\right)'v + \frac{1}{\tilde{c}\tilde{c}'}\right)\frac{\partial}{\partial v}\right]$
	$V_0^7 = \mathrm{e}^{-t}\left[\frac{2\tilde{c}}{\tilde{c}'}\frac{\partial}{\partial x} + 2\left(1 - \left(\frac{\tilde{c}}{\tilde{c}'}\right)'\right)\frac{\partial}{\partial t} + \left(\left(2 - \left(\frac{\tilde{c}}{\tilde{c}'}\right)'\right)u + \frac{\tilde{c}}{\tilde{c}'}v\right)\frac{\partial}{\partial u} - \left(\left(\frac{\tilde{c}}{\tilde{c}'}\right)'v - \frac{1}{\tilde{c}\tilde{c}'}u\right)\frac{\partial}{\partial v}\right]$

Let $u = D_1 v$. Then v satisfies a perturbed evolution equation $T(x, v, \epsilon)$ given by

$$D_2 v = f(x, v_{(1)}, \ldots, v_{(k)}) + \epsilon g(x, v_{(1)}, \ldots, v_{(k)}). \tag{11}$$

The following theorem establishes a one-to-one correspondence between approximate Lie point symmetries of $T\{x, v, \epsilon\}$ and approximate Lie point symmetries of $S\{x, u, v, \epsilon\}$.

Theorem 1. *An approximate Lie point symmetry of* $S\{x, u, v, \epsilon\}$ *induces an approximate Lie point symmetry of* $T\{x, v, \epsilon\}$ *and conversely an approximate Lie point symmetry of* $T\{x, v, \epsilon\}$ *induces an approximate Lie point symmetry of* $S\{x, u, v, \epsilon\}$.

Theorem 2. *Suppose that* $V = V_0 + \epsilon V_1$ *is an approximate potential symmetry of equation (1) and* $(\xi_{01}, \xi_{02}, \phi_0)$ *depend on* v *explicitly. Then* V_0 *is a potential symmetry of the unperturbed equation corresponding to (1).*

3. A perturbed wave equation for an inhomogeneous medium

A perturbed wave equation which describes wave propagation with variable speed in an inhomogeneous medium is of the form

$$u_{tt} + \epsilon u_t = c^2(x)u_{xx}. \tag{12}$$

Lie point symmetries, potential symmetries and exact solutions of the unperturbed wave equation of (12)

$$u_{tt} = c^2(x)u_{xx} \tag{13}$$

have been discussed in [6, 12, 13]. The auxiliary system of (13) is

$$\begin{aligned} u_x &= v_t \\ u_t &= c^2(x)v_x. \end{aligned} \tag{14}$$

Equation (13) admits potential symmetries if and only if the wave speed $c(x)$ satisfies

$$cc'\left(\frac{c}{c'}\right)'' = \mu \qquad \mu = \text{const.} \tag{15}$$

The potential symmetries of (13) are listed in table 1.

In table 1, $\tilde{c}(x)$ satisfies (15), with $\mu \neq 0$, which cannot be solved explicitly but reduces to one of the following first-order ODEs:

$$\begin{aligned} c' &= \nu^{-1}\sin(\nu \ln c) \\ c' &= \nu^{-1}\sinh(\nu \ln c) \\ c' &= \ln c \\ c' &= \nu^{-1}\cosh(\nu \ln c). \end{aligned} \tag{16}$$

For $\mu = 0$, to within arbitrary scalings and translations in x, equation (15) implies

$$c(x) = \mathrm{e}^x \qquad \text{or} \qquad c(x) = x^a \tag{17}$$

where a is an arbitrary constant.

In this section, we are concerned with the approximate potential symmetries of equation (12), namely under what conditions equation (12) admits nontrivial approximate potential symmetries. To this end, we form an associated system

$$\begin{aligned} u_t + \epsilon u &= c^2(x) v_x \\ u_x &= v_t. \end{aligned} \tag{18}$$

From theorem 2, we notice that the approximate potential symmetries of (12) are inherited from the potential symmetries of (13). We now show by means of one example, namely, V_0^5, how one can find approximate potential symmetries of (12). Suppose that (18), with $c(x) = \mathrm{e}^x$, admits an approximate Lie point symmetry

$$V^5 = V_0^5 + \epsilon V_1^5 \tag{19}$$

where

$$V_1^5 = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \psi(x,t,u,v)\frac{\partial}{\partial v}. \tag{20}$$

One can determine V_1^5 by the three-step algorithm. The $H = (H_1, H_2)$ function is given by

$$\begin{aligned} H_1 &= \frac{1}{\epsilon} V_0^{5[1]}(u_x - v_t)|_{\{u_x = v_t, u_t + \epsilon u = \mathrm{e}^{2x} v_x\}} = 0 \\ H_2 &= \frac{1}{\epsilon} V_0^{5[1]}(u_t + \epsilon u - \mathrm{e}^{2x} v_x)|_{\{u_x = v_t, u_t + \epsilon u = \mathrm{e}^{2x} v_x\}} = 2v + 4tu. \end{aligned} \tag{21}$$

Then from the invariance condition (9) for (18), namely

$$\begin{aligned} &V_1^{5[1]}(u_x - v_t)|_{\{u_x = v_t, u_t = \mathrm{e}^{2x} v_x\}} = 0 \\ &V_1^{5[1]}(u_t - \mathrm{e}^{2x} v_x)|_{\{u_x = v_t, u_t = \mathrm{e}^{2x} v_x\}} + 2v + 4tu = 0 \end{aligned} \tag{22}$$

we deduce that ξ, τ, ϕ and ψ satisfy the relations

$$\begin{aligned} &\xi_u = \xi_v = \tau_u = \tau_v = 0 \\ &\phi = f_1(x,t)u + g_1(x,t)v \\ &\psi = f_2(x,t)u + g_2(x,t)v \end{aligned} \tag{23}$$

so that ξ, τ, f_i and g_i, $i = 1, 2$ obey the following overdetermined system:

$$\begin{aligned} f_1 &= g_2 + \xi \\ g_1 &= \mathrm{e}^{2x} f_2 \\ f_{1t} &= \mathrm{e}^{2x} f_{2x} - 4t \\ g_{1t} &= \mathrm{e}^{2x} g_{2x} - 2 \\ f_{2t} &= f_{1x} \\ g_{2t} &= g_{1x} \\ \tau_t &= \xi_x - \xi \\ \xi_t &= \mathrm{e}^{2x} \tau_x. \end{aligned} \tag{24}$$

Table 2. Approximate potential symmetries of equation (12).

$c(x)$	Generators
x^a $(a \neq 1)$	$V^1 = V_0^1 + \frac{a-1}{2a}\epsilon\left[(tu+xv)\frac{\partial}{\partial u} + (x^{1-2a}u+tv)\frac{\partial}{\partial v}\right]$
	$V^2 = V_0^2 + \epsilon\Bigg[\left(\frac{3}{2a}tx^{2-2a} + \frac{(1-a)^2}{2a}t^3\right)\frac{\partial}{\partial t} + \left(\frac{1}{2a(1-a)}x^{3-2a} + \frac{3(1-a)}{2a}t^2x\right)\frac{\partial}{\partial x}$ $+\left(\left(-\frac{5}{4a}x^{2-2a} + \frac{(1-a)(6a-5)}{4a}t^2\right)u - \frac{5(1-a)}{2a}xtv\right)\frac{\partial}{\partial u}$ $+\left(-\frac{5(1-a)}{2a}tx^{1-2a}u + \left(\frac{3a-5}{4a(1-a)}x^{2-2a} - \frac{5(1-a)}{4a}t^2\right)v\right)\frac{\partial}{\partial v}\Bigg]$
x	$V^3 = V_0^3 + \epsilon\left[t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} - v\ln x\frac{\partial}{\partial v}\right]$
e^x	$V^4 = V_0^4 + \epsilon\left[\frac{1}{2}v\frac{\partial}{\partial u} + \frac{1}{2}e^{-2x}u\frac{\partial}{\partial v}\right]$,
	$V^5 = V_0^5 + \epsilon\Bigg[(e^{-2x}+3t^2)\frac{\partial}{\partial x} - (3te^{-2x}+t^3)\frac{\partial}{\partial t}$ $+\left(\left(\frac{5}{2}e^{-2x}+3t^2\right)u - 5tv\right)\frac{\partial}{\partial u} + \left(\frac{3}{2}e^{-2x}v - 5te^{-2x}u\right)\frac{\partial}{\partial v}\Bigg]$

The solution of the above system (24) gives rise to

$$\begin{aligned}
\xi &= e^{-2x} + 3t^2 + c_3t + c_4 \\
\tau &= -3e^{-2x}t - t^3 - \tfrac{1}{2}c_3(t^2 + e^{-2x}) - c_4t + c_5 \\
f_1 &= \tfrac{5}{2}e^{-2x} + 3t^2 + c_3t + c_1 + c_4 \\
f_2 &= -5te^{-2x} - \tfrac{1}{2}c_3e^{-2x} \\
g_1 &= -5t - \tfrac{1}{2}c_3 \\
g_2 &= \tfrac{3}{2}e^{-2x} + c_1.
\end{aligned} \tag{25}$$

Hence, we obtain a nontrivial approximate potential symmetry V^5 of equation (12), with wave speed $c(x) = e^x$. This symmetry is given in table 2. In the same way, we can deal with the other cases listed in table 1. For $c(x)$ satisfying (16), equation (12) does not admit nontrivial approximate potential symmetries. In fact we have established theorem 3.

Theorem 3. *Equation (12), with respect to the auxiliary system (18), admits nontrivial approximate potential symmetries if and only if $c(x)$ is given by (17).*

Approximate potential symmetries of (12) are listed in table 2.

4. Nonlinear diffusion equation with a perturbed convection term

In this section, we consider the nonlinear diffusion equation with a perturbed convection term

$$u_t = (D(u)u_x)_x + \epsilon(K(u))_x. \tag{26}$$

The unperturbed equation of (26) is

$$u_t = (D(u)u_x)_x \tag{27}$$

and the associated auxiliary system of (27) is

$$\begin{aligned}
u &= v_x \\
v_t &= D(u)u_x.
\end{aligned} \tag{28}$$

Table 3. Potential symmetries of equation (27).

$D(u)$	Generators
Arbitrary	$X_0^1 = \frac{\partial}{\partial v}, X_0^2 = \frac{\partial}{\partial t}, X_0^3 = \frac{\partial}{\partial x}, X_0^4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$
$u^{\nu} (\nu \neq -2)$	$X_0^1, X_0^2, X_0^3, X_0^4, X_0^5 = x\frac{\partial}{\partial x} + \frac{2}{\nu}u\frac{\partial}{\partial u} + \left(1 + \frac{2}{\nu}\right)v\frac{\partial}{\partial v}$
u^{-2}	$X_0^1, X_0^2, X_0^3, X_0^4, X_0^6 = -xv\frac{\partial}{\partial x} + u(v + xu)\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v},$
	$X_0^7 = -x(v^2 + 2t)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} + u(v^2 + 6t + 2xuv)\frac{\partial}{\partial u} + 4tv\frac{\partial}{\partial v},$
	$X_0^{\infty} = \theta(v, t)\frac{\partial}{\partial x} - u^2\frac{\partial \theta}{\partial v}\frac{\partial}{\partial u}$
$\frac{1}{u^2 + pu + q}\exp\left[r\int\frac{\mathrm{d}u}{u^2 + pu + q}\right]$	$X_0^1, X_0^2, X_0^3, X_0^4,$
	$X_0^8 = v\frac{\partial}{\partial x} + (r - p)t\frac{\partial}{\partial t} - (u^2 + pu + q)\frac{\partial}{\partial u} - (qx + pv)\frac{\partial}{\partial v}$

The potential symmetries of (27) were discussed in [6, 10] and are listed in table 3.

The auxiliary system of (26) is

$$\begin{aligned} u &= v_x \\ v_t &= D(u)u_x + \epsilon K(u). \end{aligned} \tag{29}$$

We now show how one can determine approximate potential symmetries of (26) that correspond to the symmetries of table 3.

For $D = u^{-2}$, a nontrivial potential symmetry of (27) is X_0^7. A direct calculation gives

$$\begin{aligned} &X_0^{7[1]}(v_t - u^{-2}u_x - \epsilon K(u))|_{\{u=v_x, v_t=u^{-2}u_x+\epsilon K(u)\}} \\ &\quad = \epsilon[(2xuv - 4t)K - (6tu + uv^2 + 2xu^2v)K']. \end{aligned}$$

So the H-function is

$$H = (0, (2xuv - 4t)K - (6tu + uv^2 + 2xu^2v)K'). \tag{30}$$

Let X_1^7 be an operator of the form (20). The infinitesimal criterion for invariance implies that ξ, τ, ϕ, ψ satisfy the following system:

$$\begin{aligned} &\tau_x = \tau_u = \tau_v = \psi_u = 0 \\ &\phi = (\tfrac{1}{2}\tau_t - \xi_x)u - \xi_v u^2 \\ &\psi_v - \phi_u - \tau_t + \xi_x + \frac{2}{u}\phi = 0 \\ &\psi_t - \xi_t u - (4t - 2xuv)K - (6tu + uv^2 + 2xu^2v)K' - u^{-2}(\phi_x + u\phi_v) = 0. \end{aligned} \tag{31}$$

The solution of system (31) yields

$$\begin{aligned} \tau &= -2d_1t^2 + d_2t + d_3 \\ \xi &= \frac{\alpha}{4}x^2v^2 + \frac{1}{2}d_1xv^2 + (d_6x + d_7)v + \frac{\alpha}{2}x^2t + d_1tx - \tfrac{5}{2}\beta t^2 + d_4x + d_8 + \int^v\int^v p(s, t)\,\mathrm{d}s\,\mathrm{d}v \\ \phi &= -2d_1tv + \frac{d_2}{2}v - 2d_6t + d_8 \\ \psi &= \left(-\frac{\alpha}{2}xv^2 - \alpha xt - 3d_1t - \frac{1}{2}d_1v^2 - d_6v - d_4 + \frac{1}{2}d_2\right)u \\ &\qquad -\left(\tfrac{1}{2}\alpha x^2v + d_1xv + d_6x + \int^v p(s, t)\,\mathrm{d}s + d_{10}\right)u^2 \\ K(u) &= \frac{\beta}{2}u - \frac{\alpha}{2}u^{-1} \end{aligned} \tag{32}$$

Table 4. Approximate potential symmetries of equation (26).

$D(u)$	$K(u)$	Generators
u^{ν} $(\nu \neq -2)$	$K_1(u)$	$X^5 = X_0^5 + \epsilon\left[\left(\frac{1}{2}x^2 + \alpha t\right)\frac{\partial}{\partial x} + \left(\frac{\nu+2}{\nu}v + \frac{2}{\nu}xu\right)\frac{\partial}{\partial u} + \left(\frac{\nu+2}{\nu}xv + \beta t\right)\frac{\partial}{\partial v}\right]$
u^{-2}	$\alpha u^{-1} - \beta \ln u + \gamma u$	$X^4 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + \epsilon\left[\frac{1}{2}(\alpha x^2 + \beta x v)\frac{\partial}{\partial x} - \left(\left(\alpha x + \frac{1}{2}\beta v\right)u + \frac{1}{2}\beta x u^2\right)\frac{\partial}{\partial u}\right]$
u^{-2}	$\frac{1}{2}\beta u - \frac{1}{2}\alpha u^{-1}$	$X^7 = X_0^7 + \epsilon\left[\left(\frac{1}{4}\alpha(x^2v^2 + 2x^2t) - \frac{1}{2}\beta(tv^2 + 5t^2)\right)\frac{\partial}{\partial x} - \left(\frac{1}{2}\alpha(xv^2 + 2xt)u + \left(\frac{\alpha}{2}x^2v - \beta t v\right)u^2\right)\frac{\partial}{\partial u}\right]$
u^{-2}	$\beta u - \alpha u^{-1}$	$X^6 = X_0^6 + \epsilon\left[\left(\frac{1}{2}\alpha x^2 v - \beta t v\right)\frac{\partial}{\partial x} - \left(\alpha x u v + \left(\frac{1}{2}\alpha x^2 - \beta t\right)u^2\right)\frac{\partial}{\partial u}\right]$
$\frac{1}{u^2 + pu + q}\exp\left[r\int\frac{\mathrm{d}u}{u^2 + pu + q}\right]$	$K_2(u)$	$X^8 = X_0^8 + \epsilon\left(\alpha\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial v}\right)$

where $p(v,t) = q(v,t) - \beta t$ and q satisfies the heat equation

$$q_t = q_{vv}. \tag{33}$$

Thus, we obtain the nontrivial approximate potential symmetry X^7 of the equation

$$u_t = (u^{-2}u_x)_x + \tfrac{1}{2}\epsilon(\beta u - \alpha u^{-1})_x \tag{34}$$

which is given in table 4. In a similar fashion, we can determine all the approximate potential symmetries of (26). These are listed in table 4.

In table 4, $K_1(u)$ is given by

$$K_1(u) = \gamma u^{\frac{\nu+2}{2}} - \left(1 + \frac{4}{\nu} - \alpha\right)u - \frac{\nu}{\nu+2}\beta$$

and $K_2(u)$ satisfies

$$(u^2 + pu + q)K_2' - (u + r)K_2 = \alpha u + \beta \tag{35}$$

which has the solution

$$
K_2(u) = \begin{cases}
(u^2 + pu + q)^{\frac{1}{2}} \left(\dfrac{2u + p + \sqrt{p^2 - 4q}}{2u + p - \sqrt{p^2 - 4q}} \right)^{\frac{p-2r}{2\sqrt{p^2-4q}}} \\
\times \left[\mu + \displaystyle\int^u \dfrac{\alpha u + \beta}{(u^2 + pu + q)^{\frac{3}{2}}} \right. \\
\left. \times \left(\dfrac{2u + p - \sqrt{p^2 - 4q}}{2u + p + \sqrt{p^2 - 4q}} \right)^{\frac{p-2r}{2\sqrt{p^2-4q}}} \mathrm{d}u \right] & \text{if} \quad p^2 > 4q \\
(u^2 + pu + q)^{\frac{1}{2}} \exp \left[\dfrac{2r - p}{\sqrt{4q - p^2}} \arctan \dfrac{2u + p}{\sqrt{4q - p^2}} \right] \\
\times \left[\mu + \displaystyle\int^u \dfrac{\alpha u + \beta}{(u^2 + pu + q)^{\frac{3}{2}}} \right. \\
\left. \times \exp \left[\dfrac{p - 2r}{\sqrt{4q - p^2}} \arctan \dfrac{2u + p}{\sqrt{4q - p^2}} \right] \mathrm{d}u \right] & \text{if} \quad p^2 < 4q \\
\left(u + \dfrac{p}{2} \right) \exp \left(\dfrac{\frac{p}{2} - r}{u + \frac{p}{2}} \right) \left[\mu + \displaystyle\int^u \dfrac{\alpha u + \beta}{(u + \frac{p}{2})^3} \exp \left(\dfrac{r - \frac{p}{2}}{u + \frac{p}{2}} \right) \mathrm{d}u \right] & \text{if} \quad p^2 = 4q.
\end{cases}
$$

5. Approximate invariant solutions of (12) and (26)

In this section, we utilize the approximate potential symmetries obtained in the previous sections to obtain some approximate, up to first order in ϵ, invariant solutions of equations (12) and (26). To place this in proper perspective, we remark on some of the approaches that could be used. The standard perturbation method involves the straightforward expansion of the dependent variables $u^\alpha = v^\alpha + \epsilon w^\alpha + o(\epsilon^2)$, which is inserted into the perturbed differential equation system (ϵ is a small parameter) $E_0^\beta(x, u, \ldots, u_{(k)}) + \epsilon E_1^\beta(x, u, \ldots, u_{(k)}) = 0$, $\beta = 1, \ldots, r$. This in general, after separation with respect to ϵ, results in a coupled system of differential equations that requires solutions for v^α and w^α which quite often are difficult to achieve. A symmetry approach was used in [15] to obtain solutions of the coupled system corresponding to perturbed nonlinear wave equations. In [15], symmetries of the coupled system were referred to as first-order approximate symmetries of the original perturbed equation. Another approach, which we alluded to earlier, is that of [2,3,5], wherein the authors utilize approximate symmetry generators of the perturbed equation to construct approximate invariant solutions of the equation. Both of these symmetry approaches are complementary to each other in the sense that the symmetries of the coupled system and the approximate symmetries of the original perturbed equation can be different from one another. Another recent method is that of approximate conditional symmetries [21], which may also provide new approximate solutions. Our approach here is also symmetry based and is complementary to both the approaches of [2–5] and [15] as well as that of [21] since approximate potential symmetries cannot be obtained by these symmetry methods and our method need not yield the symmetries that can arise in the approaches of [2, 3, 5] and [15] as well as that of [21]. We shall further illustrate this point in example 1. The solutions obtained in the following examples 1–4 cannot be derived by the methods cited as they are constructed via approximate potential symmetries which in essence are nonlocal symmetries of the original equation.

Example 1. Equation

$$u_{tt} + \epsilon u_t = \mathrm{e}^{2x} u_{xx} \tag{36}$$

admits the approximate potential symmetry V^4 of table 2. The characteristic equations for V^4 are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}u}{\frac{1}{2}\epsilon v} = \frac{\mathrm{d}v}{\frac{1}{2}\epsilon \mathrm{e}^{-2x}u - v}. \tag{37}$$

Solving the first equation of (37), we obtain an invariant of V^4, namely $\lambda = t\mathrm{e}^x$. The last two equations of (37) then become

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\epsilon v}{2t} \tag{38}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{t}\left(\frac{1}{2}\epsilon t\lambda^{-2}u - v\right). \tag{39}$$

The elimination of v from (38) and (39) results in u satisfying

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \frac{1}{4}\epsilon^2\lambda^{-2}u = 0$$

which gives

$$u = \exp[\tfrac{1}{2}\epsilon\lambda^{-1}t]u_1(\lambda) + \exp[-\tfrac{1}{2}\epsilon\lambda^{-1}t]u_2(\lambda).$$

Hence, the first-order in ϵ, approximate solution u is

$$u = A(\lambda) + \epsilon B(\lambda)\mathrm{e}^{-x} \qquad \lambda = t\mathrm{e}^x. \tag{40}$$

The insertion of (40) into (36) yields that $A(\lambda)$ and $B(\lambda)$ satisfy the system

$$\begin{aligned}&(1-\lambda^2)A'' = \lambda A'\\ &(1-\lambda^2)B'' + \lambda B' - B = -A'\end{aligned} \tag{41}$$

which has the general solution

$$\begin{aligned}A(\lambda) &= c_1 + c_2 \operatorname{arccosh}\lambda\\ B(\lambda) &= c_3\lambda + c_4\left(\lambda \operatorname{arccosh}\lambda - \sqrt{\lambda^2-1}\right) - \tfrac{1}{2}c_2\ln(\lambda^2-1)\left(\lambda\operatorname{arccosh}\lambda - \sqrt{\lambda^2-1}\right)\\ &\quad -c_2\lambda\left(\operatorname{arccosh}\lambda - \int_0^{\ln\sqrt{\lambda^2-1}} \operatorname{arcsinh}\mathrm{e}^y\,\mathrm{d}y\right)\end{aligned} \tag{42}$$

where here and hereafter c_i, $i = 1, 2, \ldots,$ are arbitrary constants.

For this example, we comment on the two symmetry approaches [2, 3, 5] and [15] as well as ours. Using the method of [2, 3, 5], an approximate symmetry of (36) is

$$Y = \left[-t + \epsilon\left(\frac{1}{2}t^2 + \frac{1}{2}\mathrm{e}^{-2x}\right)\right]\frac{\partial}{\partial t} + [1-\epsilon t]\frac{\partial}{\partial x}$$

which does not arise in the approach used here. Also the method employed in [15] results in the coupled system (substitute $u = v + \epsilon w$ into (36) and split with respect to ϵ)

$$\begin{aligned}&v_{tt} - \mathrm{e}^{2x}v_{xx} = 0\\ &w_{tt} + v_t = \mathrm{e}^{2x}w_{xx}.\end{aligned} \tag{43}$$

The nontrivial point symmetries of (43) do not provide new information from those symmetries that are obtainable using the method of [2, 3, 5] (see [16] for the link between the approaches [2, 3, 5] and [15]). However, in general, they are complementary [16]. Notwithstanding, one cannot obtain the approximate potential symmetry V^4 by invoking the results of [2, 3, 5] or [15] as V^4 of table 2 is nonlocal.

Example 2. Equation (36) also admits the approximate potential symmetry V^5. A group-invariant solution of the unperturbed equation

$$u_{tt} = \mathrm{e}^{2x} u_{xx} \tag{44}$$

corresponding to V_0^5 is

$$u_0 = c_1 t (t^2 - \mathrm{e}^{-2x})^{-\frac{3}{2}} + c_2 (t^2 - \mathrm{e}^{-2x})^{-\frac{1}{2}}. \tag{45}$$

An approximate solution of order ϵ to (36) is

$$u = u_0 + \epsilon u_1(x, \lambda) \qquad \lambda = t^2 \mathrm{e}^x - \mathrm{e}^{-x} \tag{46}$$

with u_1 satisfying the following linear second-order ODE with parameter λ:

$$\begin{aligned}(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})u_1'' &- 2(\lambda\mathrm{e}^{-x} + 2\mathrm{e}^{-2x})u_1' + (\lambda\mathrm{e}^{-x} + \tfrac{3}{2}\mathrm{e}^{-2x})u_1 \\ &= 2(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})g_x - (\lambda\mathrm{e}^{-x} + 2\mathrm{e}^{-2x})g - (\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})^{\frac{1}{2}} f\end{aligned} \tag{47}$$

where

$$\begin{aligned}
g(x,\lambda) &= -\frac{1}{\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x}}[3\mathrm{e}^{-2x}(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})\tilde{u}_0 - (6\mathrm{e}^{-2x} + 7\lambda\mathrm{e}^{-x})v_0] \\
f(x,\lambda) &= \frac{3\lambda\mathrm{e}^{-3x} + 2\mathrm{e}^{-4x}}{8(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})}\tilde{u}_0 - \frac{3}{8}\mathrm{e}^{-2x}(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})^{-\frac{1}{2}} v_0, \\
\tilde{u}_0 &= c_1(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})^{\frac{1}{2}}\mathrm{e}^{\frac{3}{2}x}\lambda^{-\frac{3}{2}} + c_2\lambda^{-\frac{1}{2}}\mathrm{e}^{\frac{1}{2}x}, \\
v_0 &= c_1\lambda^{-\frac{3}{2}}\mathrm{e}^{-\frac{1}{2}x} + c_2\lambda^{-\frac{1}{2}}\mathrm{e}^{\frac{1}{2}x}(\lambda\mathrm{e}^{-x} + \mathrm{e}^{-2x})^{\frac{1}{2}}.
\end{aligned}$$

Example 3. The nonlinear diffusion equation

$$u_t = (u^{-2}u_x)_x + \epsilon(\beta u - \alpha u^{-1})_x \tag{48}$$

admits the approximate potential symmetry X^6 of table 4. We now find an approximate invariant solution of the first order of precision of (48)

$$u = u_0 + \epsilon u_1 \qquad v = v_0 + \epsilon v_1. \tag{49}$$

The subsidiary equations for X^6 are

$$\begin{aligned}\frac{\mathrm{d}t}{0} &= \frac{\mathrm{d}x}{-xv + \epsilon(\frac{1}{2}\alpha x^2 v - \beta t v)} \\ &= \frac{\mathrm{d}u}{u(v + xu) - \epsilon[\alpha x u v + (\frac{1}{2}\alpha x^2 - \beta t)u^2])} = \frac{\mathrm{d}v}{2t}.\end{aligned} \tag{50}$$

It is easy to find an exact solution of the unperturbed equation of (48) corresponding to X_0^6. This is

$$u_0 = \frac{2t}{x}[c_1 t - 2t\ln t - 4t\ln x]^{-\frac{1}{2}}. \tag{51}$$

From (50), we obtain u_1, v_1 and an approximate invariant solution of (48)

$$\begin{aligned}u = \frac{2t}{x}&[c_1 t - 2t\ln t - 4t\ln x]^{-\frac{1}{2}} \\ &+\epsilon\left[\frac{2\beta t^2 - \alpha t x^3}{x^2(c_1 t - 2t\ln t - 4t\ln x)^{\frac{1}{2}}} + \frac{2t((t + t_0)x - \frac{1}{2}\alpha t x^3 - 2\beta t^2)}{x^2(c_1 t - 2t\ln t - 4t\ln x)^{\frac{3}{2}}}\right].\end{aligned}$$

Example 4. The perturbed Mullins equation [22]

$$u_t = \left(\frac{u_x}{1+u^2}\right)_x + \epsilon(\alpha(u^2+1)^{\frac{1}{2}} + u)_x \tag{52}$$

admits the approximate potential symmetry

$$X_4 = v\frac{\partial}{\partial x} - (u^2+1)\frac{\partial}{\partial u} - (x+\epsilon t)\frac{\partial}{\partial v} \tag{53}$$

which gives an approximate invariant solution of order ϵ of (52)

$$u = x[2(t_0 - t) - x^2]^{-\frac{1}{2}} + \epsilon(\alpha(t_0 - t)x - 2t(t_0 - t))(2(t_0 - t) - x^2)^{-\frac{3}{2}} \tag{54}$$

where t_0 is a positive constant. The solution (54) is well defined for $t + \frac{x^2}{2} \leqslant t_0$.

6. Conclusion

We have presented the approximate potential symmetry method for partial differential equations. A complete classification of the approximate potential symmetries, with respect to the auxiliary systems used, for a perturbed wave equation with a variable wave speed and a nonlinear diffusion equation with convection terms was presented. As applications, the method was used to obtain approximate invariant solutions of order ϵ for several examples. These solutions cannot be obtained by the usual approximate symmetry method [2, 3, 5] or the approach used in [15]. The method adopted here is complementary to those of [2, 3, 5] and [15] as well as that of [21]. Our method can also be used to find possible new approximate potential symmetries for a given nonlinear partial differential equation with respect to other conservation laws.

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